Periods and Eisenstein series

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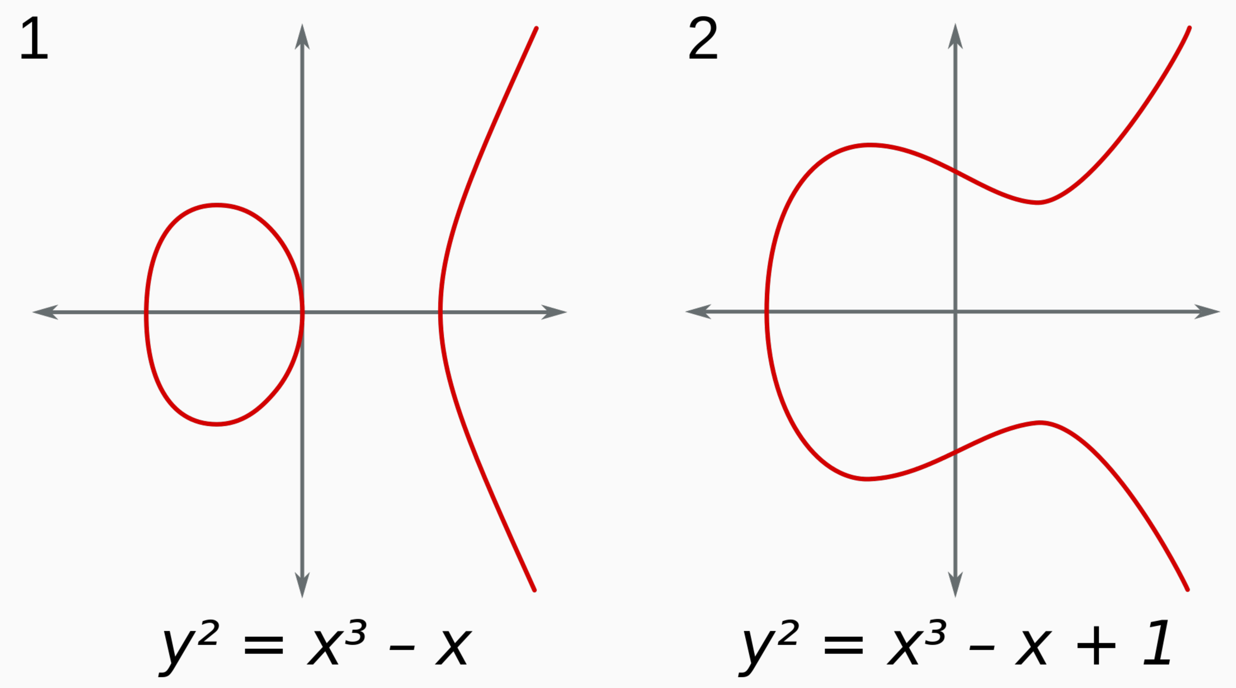
Elliptic curves and L-functions

The theory of elliptic curves and their L-functions play an central role in contemporary number theory. This is because elliptic curves are some of the simplest algebraic curves, with nonetheless difficult problems surrounding them.
An elliptic curve is a curve

$$E : y^2 = x^3 + ax + b$$

such that it has no self intersections or cusps ("spikes").



By the modularity theorem, there is an unique cusp form $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$ associated to the curve, where $q = e^{2\pi i \tau}$. This is a function such that for almost all primes p, $a_p = p - N_p$, where N_p is the number of points on E modulo p. The function satisfies many special properties, namely

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau + d)^k f$$

For many choices of a, b, c, d.
The L-function of the elliptic curve is defined in terms of the cusp form through $L(E, s) = C \int_0^{\infty} f(i\tau)\tau^{s-1} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, where C is $\frac{\pi^s}{(s-1)}!$ if s is an integer. If s is an integer, we also call $L(E, s)$ an L-value.

Beilinsons conjecture

An important conjecture by Beilinson concerning L-values gives a formula that Mahler measure of any polynomial equals an L-value $L(E, k)$ for some elliptic curve E and integer k.
Here the Mahler measure of a polynomial P is defined as

$$m(P) = \frac{1}{(2\pi i)^k} \int \cdots \int_{|x_1|=\cdots=|x_k|=1} \log|P(x_1, \ldots, x_k)| \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_k}{x_k}.$$

This can be rewritten as an integral of algebraic functions. A constant that can be written in such a way is called a period. It is known that L-values are periods. This could be utilized to prove the equality between a Mahler measure and an L-value, as it is expected that two equal periods can be identified using only basic calculus tools. However, even though L-values are periods, there is no practical method of writing them as periods known. My research is focused on finding such a method.

Eisenstein series

The main idea behind the method is to write the cusp form f associated to E as a product of two Eisenstein series. Using some changes of variable, it follows that the L-function of f ($L(f, k) = L(E, k)$) equals the L function of a product different Eisenstein series times an algebraic constant.
Here Eisenstein series are defined (for $k > 2$) as

$$E_{a,b}^{N,k}(\tau) = \beta_k \sum_{\substack{m \equiv a \\ n \equiv b}\ \mathrm{mod}\ N}{}' (mN\tau + n)^{-k}$$

for with $\beta_k = \frac{(k-1)!}{(2\pi i)^k}$. In the classical case of $N = 1$ this reduces to the simpler expression

$$E_k(\tau) = \beta_k \sum_{m,n \in \mathbb{Z}}{}' (m\tau + n)^{-k}.$$

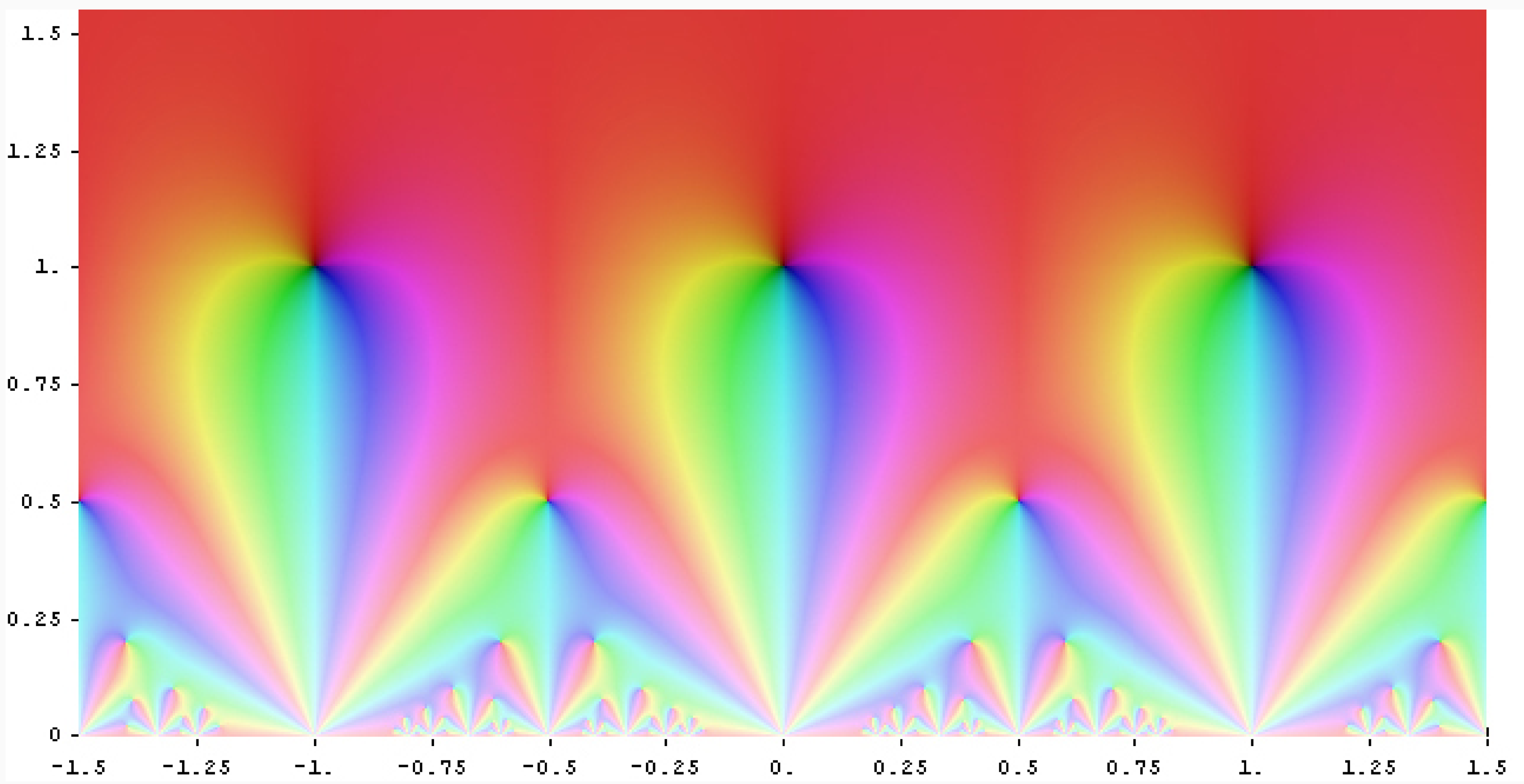


Figure: A colour wheel graph of the Eisenstein series E_6.

Between Eisenstein series, special (algebraic) relations exist, and using these, it is possible to write these series as an algebraic function of a special function $x(\tau)$, times an function defined as an integral of a function of $x(\tau)$. So, by finding such a function x, and expressing the Eisenstein series in terms of it, the L-value $L(E, k)$ can be written as a period explicitly. Using this method I have found period expressions that were not known yet.
As an example for the elliptic curve (depicted on the top left of this poster)

$$E : y^2 = x^3 - x,$$

we have

$$L(E, 4) = \frac{\pi^3}{1536} \times \int \cdots \int_{[0,1]^4} \frac{(1 - 6y + y^2)\sqrt{1-y}\,\mathrm{d}y\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3}{\sqrt{y(1+y)(y^4 + 4(1-y^2)(1-y_1^2)(1-y_2^2)(1-y_3^2))}}.$$

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